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On the energetic balance for the flow of an Oldroyd-B fluid due to a flat plate subject to a time-dependent shear stress

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ABSTRACT

Exact and approximate expressions for the power due to the shear stress at the wall L, the dissipation Φ and the boundary layer thickness δ are established for the unsteady flow of an Oldroyd-B fluid driven by the transverse motion of an infinite plate subject to a time-dependent shear stress. The change of the kinetic energy with time is also obtained from the energetic balance. Similar expressions for Newtonian, Maxwell and second-grade fluids are obtained as limiting cases of general results. Series solutions for the velocity and shear stress are also obtained for small values of the dimensionless relaxations and retardation times. Graphical illustrations corresponding to the exact expressions for L, Φ and δ agree with the associated asymptotic approximations. Usually for many industrial applications the velocity of the wall is given and what is required is the energy that is necessary to keep the wall running with the prescribed value. The problem discussed by us now is that where, on the contrary, the wall shear stress is given but the velocity and the energy of the medium are required.

1. Introduction

Studies of different flows of Newtonian or non-Newtonian fluids have received much attention, being important both from theoretical and experimental points of view. Exact or approximate solutions are well-known in the literature (see for instance [1–13] for the flow over an infinitely extended plate), while energetic aspects as regards these motions are scarcely met. The first notable results of this kind seem to be those of Bühler and Zierep [14] concerning the Rayleigh–Stokes problem for Newtonian fluids. These results have been recently extended to second-grade and Maxwell fluids [15,16]. Analogous results for the motion of the same fluids due to an infinite plate subject to a constant shear stress or induced by a constantly accelerating plate have also been established in [17,18].

The main purpose of this work is to provide a study of the energetic balance for the unsteady flow of an Oldroyd-B fluid, driven by the transverse motion of an infinite plate subject to a time-dependent shear stress. The Oldroyd-B fluids store energy as linearized elastic solids and their dissipation is due to two dissipative mechanisms which arise from a mixture of two viscous fluids. They have been extensively used in many applications, the Oldroyd-B model being viewed as a successful model for describing the response of a subclass of polymeric liquids.

Of special interest for us is the energetic balance for the three terms: Changing of the kinetic energy with time, dissipation and power due to shear stress at the wall. The last term describes the energy input that is necessary to keep the medium running. A decisive question is whether these terms are larger or smaller than in the cases of Newtonian, Maxwell or

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second-grade fluids. In order to realize a clear and visible comparison between the four models, both exact and approximate expressions have been established for these very important entities. The asymptotic approximations, as well as graphical illustrations corresponding to the exact expressions, clearly show that these entities decrease for Oldroyd-B fluids with respect to Newtonian, Maxwell or second-grade fluids.

For completeness, our study is extended to find similar expressions for the boundary layer thickness. This very important entity was used earlier by Teipel [2] to find a series solution for the velocity field corresponding to the first problem of Stokes for second-grade fluids. Following Teipel [2], we also provide series solutions for the velocity field and the shear stress. The values of the two free constants, appearing in their expressions, are determined using the approximate solutions resulting from asymptotic approximations.

In the special cases where the retardation or the relaxation time tends to zero, all results that have been obtained reduce to those for Maxwell or second-grade fluids performing the same motion. If both relaxation and retardation times tend to zero, results similar to those for Newtonian fluids are recovered. In all cases, the boundary layer thickness δ, unlike L and Φ, does not depend of the constant shear stress f of the plate.

2. Mathematical formulation of the problem

The Cauchy stress tensor $\mathbf{T}$ corresponding to incompressible Oldroyd-B fluids is related to the fluid motion by the constitutive equations [8,10–13,19]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda\left(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T\right) = \mu\left[\mathbf{A} + \lambda_r\left(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^T\right)\right], \tag{1}$$

where $\mathbf{S}$ is the extra-stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin–Ericksen tensor, μ is the dynamic viscosity, λ and λ_r are relaxation and retardation times and the superposed dot indicates the material time derivative. This model includes, as special cases, the Maxwell model for $\lambda_r = 0$ and the linearly viscous fluid model for $\lambda = \lambda_r = 0$. For some motions, like those to be considered here, the governing equation also includes the motion equation for second-grade fluids (of course, neglecting the restriction $\lambda_r < \lambda$ [19] and setting $\lambda = 0$).

Let us now consider an incompressible Oldroyd-B fluid at rest occupying the space above an infinitely extended plate which is situated in the (x, z)-plane of a Cartesian coordinate system x, y and z. At time $t = 0^+$, a time-dependent shear stress

$$\mathbf{S}_{yx} = f\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right], \tag{2}$$

is applied to the plate in the x-direction. Owing to the shear, the fluid is gradually moved, its velocity being of the form

$$\mathbf{v} = \mathbf{v}(y, t) = u(y, t)\mathbf{i}, \tag{3}$$

where $\mathbf{i}$ denotes the unit vector along the x-coordinate direction. For this velocity field, the constraint of incompressibility is automatically satisfied and the extra-stress tensor $\mathbf{S}$ can also be assumed to be a function of y and t only. Substituting Eq. (3) into Eq. $(1)_2$ and bearing in mind the initial condition

$$\mathbf{S}(y, 0) = \mathbf{0} \quad \text{(the fluid being at rest up to the moment } t = 0), \tag{4}$$

we find that $S_{xz} = S_{yz} = S_{yy} = S_{zz} = 0$. With regard to the shear stress $\tau(y, t) = S_{yx}(y, t)$, it must satisfy the partial differential equation [10–12]

$$\tau(y, t) + \lambda\partial_t\tau(y, t) = \mu(1 + \lambda_r\partial_t)\partial_y u(y, t). \tag{5}$$

In the absence of body forces and a pressure gradient in the flow direction, the balance of linear momentum reduces to the meaningful equation

$$\partial_y\tau(y, t) = \rho\partial_t u(y, t), \tag{6}$$

where ρ is the constant density of the fluid. Eliminating $\tau(y, t)$ between Eqs. (5) and (6), we obtain the governing equation

$$(1 + \lambda\partial_t)\partial_t u(y, t) = \nu(1 + \lambda_r\partial_t)\partial_y^2 u(y, t); \quad y, t > 0, \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The appropriate initial and boundary conditions are [20]

$$u(y, 0) = \partial_t u(y, 0) = 0; \quad y > 0, \tag{8}$$

$$\mu(1 + \lambda_r\partial_t)\partial_y u(y, t) = (1 + \lambda\partial_t)\tau(y, t) = f \quad \text{at } y = 0 \text{ and } t > 0 \tag{9}$$

and

$$u(y, t), \partial_y u(y, t) \to 0 \quad \text{for } y \to \infty \text{ and } t > 0. \tag{10}$$

It is worth pointing out that for $\lambda \to 0$, the governing equation (7) reduces to that for a second-grade fluid (see for instance [7] or [9]).

The exact solutions corresponding to the general problem (7)–(10) are [20]

$$u(y,t) = \frac{f}{\mu}y - \frac{2f}{\mu\pi}\int_0^\infty \left\{1 - \frac{r_2\exp(r_1t) - r_1\exp(r_2t)}{r_2 - r_1}\cos(y\xi)\right\}\frac{1}{\xi^2}\,\mathrm{d}\xi; \quad y,\ t > 0, \tag{11}$$

$$\tau(y,t) = f\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] - \frac{2f}{\lambda\pi}\int_0^\infty \frac{\exp(r_2t) - \exp(r_1t)}{r_2 - r_1}\,\frac{\sin(y\xi)}{\xi}\,\mathrm{d}\xi; \quad y,\ t > 0, \tag{12}$$

where $r_{1,2} = \frac{-(1+\alpha\xi^2)\pm\sqrt{(1+\alpha\xi^2)^2 - 4\nu\lambda\xi^2}}{2\lambda}$ with $\alpha = \nu\lambda_r$.

By setting λ_r or $\lambda \to 0$ in Eqs. (11) and (12), similar solutions for Maxwell and second-grade fluids are immediately obtained (see Eq. (10) from [9] for the velocity field of a second-grade fluid). Furthermore, if both λ and $\lambda_r \to 0$, the solutions

$$u_N(y,t) = \frac{f}{\mu}y - \frac{2f}{\mu\pi}\int_0^\infty \left[1 - \mathrm{e}^{-\nu t\xi^2}\cos(y\xi)\right]\frac{1}{\xi^2}\,\mathrm{d}\xi; \qquad \tau_N(y,t) = f - \frac{2f}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\mathrm{e}^{-\nu t\xi^2}\,\mathrm{d}\xi, \tag{13}$$

corresponding to a Newtonian fluid are recovered. Of course, these solutions can be written in the simple forms (see Eqs. (A.1) and (A.2) from the Appendix)

$$u_N(y,t) = \frac{f}{\mu}\left[y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right) - 2\sqrt{\frac{\nu t}{\pi}}\exp\left(-\frac{y^2}{4\nu t}\right)\right], \tag{14}$$

$$\tau_N(y,t) = f\left[1 - \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] = f\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{15}$$

where $\mathrm{erf}(\cdot)$ is the error function of Gauss [21] and $\mathrm{erfc}(\cdot)$ is its complementary function.

The energetic balance for a certain volume V of fluid is given by the equality [14]

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{kin} + L + \Phi = 0, \tag{16}$$

where E_{kin} is the kinetic energy, L is the power due to the shear stress at the wall and Φ is the dissipation. The changing of the kinetic energy with time is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{kin} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{\rho}{2}\mathbf{v}^2\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}\left(\frac{\rho}{2}\mathbf{v}^2\right)\,\mathrm{d}V + \int_A \frac{\rho}{2}\mathbf{v}^2(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}A, \tag{17}$$

where A is the boundary of the flow domain, $\mathbf{n}$ is the unit vector normal to A and the dot "$\cdot$" denotes the inner product. For an infinite volume of rectangular cross-section with $x \in [0, \ell]$ and $z \in [0, 1]$, Eqs. (3), (6) and (17) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{kin} = \rho\int_V u(y,t)\frac{\partial u(y,t)}{\partial t}\,\mathrm{d}V = \ell\int_0^\infty u(y,t)\frac{\partial\tau(y,t)}{\partial y}\,\mathrm{d}y = -\ell u(0,t)\,\tau(0,t) - \ell\int_0^\infty \tau(y,t)\frac{\partial u(y,t)}{\partial y}\,\mathrm{d}y. \tag{18}$$

From Eqs. (16) and (18) it is clear that

$$L = L(t) = \ell u(0,t)\tau(0,t) \quad \text{and} \quad \Phi = \Phi(t) = \ell\int_0^\infty \tau(y,t)\frac{\partial u(y,t)}{\partial y}\,\mathrm{d}y. \tag{19}$$

The boundary layer thickness, resulting from [2], is given by the relation

$$\delta = \delta(t) = \frac{1}{u(0,t)}\int_0^\infty u(y,t)\,\mathrm{d}y, \tag{20}$$

and represents the thickness of the fluid layer moved with the plate by friction. One measure of the boundary layer thickness is the distance from the wall where the velocity of the fluid differs by one per cent from the external velocity.

3. Exact expressions for L, Φ and δ

Introducing $u(y,t)$ and $\tau(y,t)$, given by Eqs. (11) and (12), into Eqs. (19) and (20) we find the following expressions for L, Φ and δ:

$$L = -\frac{2\ell f^2}{\mu\pi}\left(1 - \mathrm{e}^{-t/\lambda}\right)\int_0^\infty \left\{1 - \frac{r_2\exp(r_1t) - r_1\exp(r_2t)}{r_2 - r_1}\right\}\frac{1}{\xi^2}\,\mathrm{d}\xi, \tag{21}$$

$$\Phi = \frac{4\ell f^2}{\mu\pi^2}\int_0^\infty\left\{\int_0^\infty\left[1-\frac{r_2\exp(r_1t)-r_1\exp(r_2t)}{r_2-r_1}\right]\frac{\sin(y\xi)}{\xi}\mathrm{d}\xi \times\int_0^\infty\left[1-\mathrm{e}^{-t/\lambda}-\frac{\exp(r_2t)-\exp(r_1t)}{\lambda(r_2-r_1)}\right]\frac{\sin(y\xi)}{\xi}\,\mathrm{d}\xi\right\}\,\mathrm{d}y, \tag{22}$$

$$\delta = -\frac{\pi}{2}\frac{\int_0^\infty\left\{y-\frac{2}{\pi}\int_0^\infty\left[1-\frac{r_2\exp(r_1t)-r_1\exp(r_2t)}{r_2-r_1}\cos(y\xi)\right]\frac{1}{\xi^2}\,\mathrm{d}\xi\right\}\,\mathrm{d}y}{\int_0^\infty\left[1-\frac{r_2\exp(r_1t)-r_1\exp(r_2t)}{r_2-r_1}\right]\frac{1}{\xi^2}\,\mathrm{d}\xi}. \tag{23}$$

Of course, on setting $\lambda_r \to 0$ or $\lambda \to 0$ in Eqs. (21)–(23), the exact expressions of L, Φ and δ for Maxwell or second-grade fluids [17] are recovered. If both λ_r and λ tend to zero, the exact expressions [17]

$$L_N = L_N(t) = -\frac{2\ell f^2}{\mu\pi}\int_0^\infty\left(1-\mathrm{e}^{-\nu t\xi^2}\right)\frac{1}{\xi^2}\,\mathrm{d}\xi, \tag{24}$$

$$\Phi_N = \Phi_N(t) = \frac{4\ell f^2}{\mu\pi^2}\int_0^\infty\left\{\int_0^\infty\left(1-\mathrm{e}^{-\nu t\xi^2}\right)\frac{\sin(y\xi)}{\xi}\mathrm{d}\xi\right\}^2\,\mathrm{d}y, \tag{25}$$

$$\delta_N = \delta_N(t) = -\frac{\pi}{2}\frac{\int_0^\infty\left\{y-\frac{2}{\pi}\int_0^\infty\left[1-\mathrm{e}^{-\nu t\xi^2}\cos(y\xi)\right]\frac{1}{\xi^2}\,\mathrm{d}\xi\right\}\,\mathrm{d}y}{\int_0^\infty\left(1-\mathrm{e}^{-\nu t\xi^2}\right)\frac{1}{\xi^2}\,\mathrm{d}\xi}, \tag{26}$$

corresponding to a Newtonian fluid, are also obtained. The above integrals can be easily evaluated (see Eqs. (A.1) and (A.3)–(A.6)) to give the simple forms

$$L_N = -\frac{2\ell f^2}{\mu}\sqrt{\frac{\nu t}{\pi}}, \qquad \Phi_N = 2\left(2-\sqrt{2}\right)\frac{\ell f^2}{\mu}\sqrt{\frac{\nu t}{\pi}}, \qquad \delta_N = \frac{\pi}{2}\sqrt{\frac{\nu t}{\pi}}. \tag{27}$$

Introducing Eq. $(27)_{1,2}$ into Eq. (16), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{kin}^N = 2\left(\sqrt{2}-1\right)\frac{\ell f^2}{\mu}\sqrt{\frac{\nu t}{\pi}}. \tag{28}$$

4. Asymptotic approximations for λ/t and $\lambda_r/t \ll 1$

Let us now consider the special case where the dimensionless relaxation and retardation times λ/t and $\lambda_r/t = \alpha/(\nu t)$ are much less than 1. For these conditions, the terms containing $\exp(-t/\lambda)$ and $\exp(r_2t)$ from Eqs. (11), (12) and (21)–(23) can be neglected since they tend to zero faster than $(\lambda/t)^2$. Furthermore, for each λ and λ_r small enough,

$$\begin{aligned}
&\sqrt{(1+\alpha\xi^2)^2-4\nu\lambda\xi^2} = 1+(\alpha-2\nu\lambda)\xi^2+2\nu\lambda(\alpha-\nu\lambda)\xi^4+\cdots,\\
&\frac{1}{\sqrt{(1+\alpha\xi^2)^2-4\nu\lambda\xi^2}} = 1-\alpha\xi^2+2\nu\lambda\xi^2+\cdots,\\
&\mathrm{e}^{r_1t} = \mathrm{e}^{-\nu t\xi^2}\left(1+\alpha\nu t\xi^4-\nu^2\lambda t\xi^4+\cdots\right),\\
&\frac{\mathrm{e}^{r_1t}}{\lambda(r_2-r_1)} = -\mathrm{e}^{-\nu t\xi^2}\left[1+\alpha\xi^2(\nu t\xi^2-1)+\nu\lambda\xi^2(2-\nu t\xi^2)+\cdots\right],\\
&\frac{r_2\mathrm{e}^{r_1t}}{r_2-r_1} = \mathrm{e}^{-\nu t\xi^2}\left[1+\alpha\nu t\xi^4+\nu\lambda\xi^2(1-\nu t\xi^2)+\cdots\right],
\end{aligned} \tag{29}$$

are valid for each ξ and t greater than zero.

Introducing Eq. $(29)_{4,5}$ into Eqs. (11) and (12), neglecting the terms of higher order in λ and α and bearing in mind Eq. (13), we find that

$$u(y,t) = u_N(y,t)+\frac{2ft}{\rho\pi}\alpha\int_0^\infty\xi^2\mathrm{e}^{-\nu t\xi^2}\cos(y\xi)\,\mathrm{d}\xi+\frac{2f}{\rho\pi}\lambda\int_0^\infty(1-\nu t\xi^2)\,\mathrm{e}^{-\nu t\xi^2}\cos(y\xi)\,\mathrm{d}\xi+\cdots \tag{30}$$

$$\tau(y,t) = \tau_N(y,t)-\frac{2f}{\pi}\alpha\int_0^\infty\xi(\nu t\xi^2-1)\,\mathrm{e}^{-\nu t\xi^2}\sin(y\xi)\,\mathrm{d}\xi-\frac{2\nu f}{\pi}\lambda\int_0^\infty\xi(2-\nu t\xi^2)\,\mathrm{e}^{-\nu t\xi^2}\sin(y\xi)\,\mathrm{d}\xi+\cdots. \tag{31}$$

Using now Eqs. (A.7)–(A.10), the velocity $u(y,t)$ and the associated shear stress $\tau(y,t)$ can be written in the suitable forms

$$u(y,t)=u_N(y,t)+\frac{f}{2\mu}\sqrt{\frac{\nu t}{\pi}}\frac{\alpha}{\nu t}\left(1-\frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right)+\frac{f}{2\mu}\sqrt{\frac{\nu t}{\pi}}\frac{\lambda}{t}\left(1+\frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right)+O(\beta^2), \tag{32}$$

$$\tau(y,t)=\tau_N(y,t)-\frac{fy}{4\sqrt{\pi\nu t}}\frac{\alpha}{\nu t}\left(1-\frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right)-\frac{fy}{4\sqrt{\pi\nu t}}\frac{\lambda}{t}\left(1+\frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right)+O(\beta^2), \tag{33}$$

where $\beta=\max(\lambda/t,\ \lambda_r/t)$. Of course, the similar solutions for second-grade fluids (see [17], Eqs. (33) and (34)), as well as those for Maxwell fluids, are obtained from Eqs. (32) and (33) for λ and $\lambda_r\to 0$, respectively.

The asymptotic approximations for L, Φ and δ can be obtained using Eq. (29) again, or by means of the approximate solutions (32) and (33). The first alternative seems to be more suitable due to the integrals that have been included in the Appendix. Consequently, introducing Eq. (29) into Eqs. (21) and (22) and taking into account Eqs. (24) and (25), we obtain the approximate expressions for L and Φ:

$$L=L_N+\frac{2\ell f^2 t}{\rho\pi}\alpha\int_0^\infty \xi^2 e^{-\nu t\xi^2}\,d\xi+\frac{2\ell f^2}{\rho\pi}\lambda\int_0^\infty(1-\nu t\xi^2)\,e^{-\nu t\xi^2}\,d\xi+\cdots, \tag{34}$$

$$\begin{aligned}\Phi\ =\ \Phi_N-\frac{4\ell f^2}{\mu\pi^2}\int_0^\infty\Bigg\{&\int_0^\infty\left(1-e^{-\nu t\xi^2}\right)\frac{\sin(y\xi)}{\xi}\,d\xi\\ &\times\left[\alpha\int_0^\infty\xi(2\nu t\xi^2-1)\,e^{-\nu t\xi^2}\sin(y\xi)\,d\xi+\nu\lambda\int_0^\infty\xi(3-2\nu t\xi^2)\,e^{-\nu t\xi^2}\sin(y\xi)\,d\xi\right]\Bigg\}\,dy.\end{aligned} \tag{35}$$

By means of Eqs. (A.3), (A.5) and (A.9)–(A.12), the integrals in Eqs. (34) and (35) can be evaluated to give the asymptotic approximations

$$\frac{L}{L_N}=1-\frac{1}{4}\frac{\alpha}{\nu t}-\frac{1}{4}\frac{\lambda}{t}+O(\beta^2);\qquad \frac{\Phi}{\Phi_N}=1-\frac{1+\sqrt{2}}{8}\frac{\alpha}{\nu t}-\frac{3-\sqrt{2}}{8}\frac{\lambda}{t}+O(\beta^2), \tag{36}$$

where $O(\beta^2)$ is a dimensionless term.

As regards δ, from Eqs. (23) and (29), it first results that

$$\delta=-\frac{\pi}{2}\frac{\int_0^\infty\left\{y-\frac{2}{\pi}\int_0^\infty\left\{1-e^{-\nu t\xi^2}\left[1+\alpha\nu t\xi^4+\lambda\nu\xi^2(1-\nu t\xi^2)+\cdots\right]\cos(y\xi)\right\}\frac{1}{\xi^2}\,d\xi\right\}dy}{\int_0^\infty\left\{1-e^{-\nu t\xi^2}\left[1+\alpha\nu t\xi^4+\lambda\nu\xi^2(1-\nu t\xi^2)+\cdots\right]\right\}\frac{1}{\xi^2}\,d\xi}. \tag{37}$$

Lengthy but straightforward computations allow us to find that (see (A.1), (A.3)–(A.5), (A.7) and (A.8))

$$\delta=\frac{\sqrt{\pi\nu t}}{2}\frac{1-\frac{\lambda}{t}+O(\beta^2)}{1-\frac{1}{4}\frac{\alpha}{\nu t}-\frac{1}{4}\frac{\lambda}{t}+O(\beta^2)}. \tag{38}$$

This last equality, together with Eq. $(27)_3$, leads to the approximation

$$\frac{\delta}{\delta_N}=1+\frac{1}{4}\frac{\alpha}{\nu t}-\frac{3}{4}\frac{\lambda}{t}+O(\beta^2). \tag{39}$$

Eqs. (16) and (36) also imply

$$\frac{d}{dt}E_{kin}=\frac{d}{dt}E_{kin}^N\left[1-\frac{1}{4\sqrt{2}}\frac{\alpha}{\nu t}-\frac{2\sqrt{2}-1}{4\sqrt{2}}\frac{\lambda}{t}+O(\beta^2)\right]. \tag{40}$$

Finally, putting λ_r or $\lambda\ \to\ 0$ into Eqs. (36), (39) and (40), we obtain similar results for Maxwell and second-grade fluids [17]. These asymptotic approximations are very important since they allow us to make comparisons among the four models. From Eq. (36), L and Φ decrease for Oldroyd-B, Maxwell and second-grade fluids in comparison with Newtonian fluids. Furthermore, L and Φ decrease for Oldroyd-B fluids in comparison with Maxwell and second-grade fluids. The boundary layer thickness δ increases for second-grade fluids and decreases for Maxwell and Oldroyd-B fluids (if $\lambda>\lambda_r$ [19]) in comparison with Newtonian fluids, while it also decreases for Oldroyd-B fluids in comparison with the second-grade fluids.

5. The analogy to the Teipel series solution

As early as 1981, Teipel [2] studied the first problem of Stokes for second-grade fluids and provided a series solution of the form

$$u(\eta) = U\left[f_0(\eta) + \frac{\alpha}{\nu t} f_1(\eta) + O(\alpha^2)\right]; \quad \eta = \frac{y}{2\sqrt{\nu t}}, \tag{41}$$

where U is the velocity of the plate, $\alpha = \alpha_1/\rho$ and α_1 is a material constant. The functions $f_0(\cdot)$ and $f_1(\cdot)$ have been determined up to an arbitrary constant, although all initial and boundary conditions have been fulfilled. In order to determine that constant, Teipel used the initial condition $\delta(0) = 0$. This condition is physically reasonable but mathematically incorrect, the series expansion (41) not being valid for $t = 0$. However, the series solution (41) is very interesting from a mathematical and physical point of view.

Consequently, following Teipel and bearing in mind the velocity field for a Newtonian fluid (see Eq. (14)), we are looking for a series solution of the form

$$u(y,t) = \frac{f}{\mu}\sqrt{\nu t}\left[f_0(\eta) + \frac{\alpha}{\nu t} f_1(\eta) + \frac{\lambda}{t} f_2(\eta) + O(\beta^2)\right]; \quad \eta = \frac{y}{2\sqrt{\nu t}}. \tag{42}$$

Introducing Eq. (42) into Eq. (7) and identifying the coefficients of terms of the same order in $\alpha/(\nu t)$ and λ/t, we obtain for the functions $f_0(\cdot), f_1(\cdot)$ and $f_2(\cdot)$ the following ordinary differential equations:

$$\begin{aligned}
&f_0''(\eta) + 2\eta f_0'(\eta) - 2f_0(\eta) = 0,\\
&f_1''(\eta) + 2\eta f_1'(\eta) + 2f_1(\eta) = \left[\eta f_0'''(\eta) + f_0''(\eta)\right]/2,\\
&f_2''(\eta) + 2\eta f_2'(\eta) + 2f_2(\eta) = \eta^2 f_0''(\eta) + \eta f_0'(\eta) - f_0(\eta).
\end{aligned} \tag{43}$$

The appropriate boundary conditions are

$$f_0'(0) = 2 \quad \text{and} \quad f_0(\eta), \qquad f_0'(\eta) \to 0 \quad \text{as } \eta \to \infty, \tag{44}$$

$$f_1'(0) = 0 \quad \text{and} \quad f_1(\eta), \qquad f_1'(\eta) \to 0 \quad \text{as } \eta \to \infty, \tag{45}$$

$$f_2'(0) = 0 \quad \text{and} \quad f_2(\eta), \qquad f_2'(\eta) \to 0 \quad \text{as } \eta \to \infty, \tag{46}$$

respectively.

The ordinary differential equation (43), subject to the conditions (44)–(46), has the solutions

$$f_0(\eta) = 2\eta\,\mathrm{erfc}(\eta) - \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\eta^2}; \qquad f_1(\eta) = \frac{1}{\sqrt{\pi}}(a - \eta^2)\,\mathrm{e}^{-\eta^2} \tag{47}$$

and

$$f_2(\eta) = \frac{1}{\sqrt{\pi}}(\eta^2 + b)\,\mathrm{e}^{-\eta^2}, \tag{48}$$

where a and b are arbitrary constants. Introducing Eqs. (47) and (48) into Eq. (42) and bearing in mind Eq. (14), we obtain

$$u(y,t) = u_N(y,t) + \frac{f}{\mu}\sqrt{\frac{\nu t}{\pi}} \exp\left(-\frac{y^2}{4\nu t}\right)\left[\left(a - \frac{y^2}{4\nu t}\right)\frac{\alpha}{\nu t} + \left(\frac{y^2}{4\nu t} + b\right)\frac{\lambda}{t}\right] + \cdots. \tag{49}$$

This solution also contains two free constants, although all initial and boundary conditions have been satisfied. Comparing Eq. (49) with Eq. (32), we see that $a = b = 1/2$.

For completeness, let us consider a similar series expansion for the shear stress $\tau(y,t)$, namely

$$\tau(y,t) = f\left[g_0(\eta) + \frac{\alpha}{\nu t} g_1(\eta) + \frac{\lambda}{t} g_2(\eta) + O(\beta^2)\right]. \tag{50}$$

Introducing Eqs. (50) and (42) into Eq. (5), in an analogous way we obtain that

$$g_0(\eta) = \frac{1}{2} f_0'(\eta); \qquad g_1(\eta) = \frac{1}{2}\left[f_1'(\eta) - \frac{\eta}{2} f_0''(\eta)\right]; \qquad g_2(\eta) = \frac{1}{2}\left[f_2'(\eta) + \eta g_0'(\eta)\right]. \tag{51}$$

Evaluating $g_0(\eta), g_1(\eta)$ and $g_2(\eta)$ and introducing the results into Eq. (50), we obtain the same approximate expression (33) for the shear stress $\tau(y,t)$.

6. Conclusions

The results of the present paper represent a natural extension of those obtained in [17] for second-grade fluids. Exact and approximate expressions for the dissipation Φ, the power of the shear stress at the wall L and the boundary layer thickness δ are established for the unsteady flow of an Oldroyd-B fluid driven by the transverse motion of an infinite plate subject to a time-dependent shear stress. The changing of the kinetic energy with time is also determined from the energetic balance. The asymptotic approximations, obtained for small values of the dimensionless relaxation and retardation times, can be

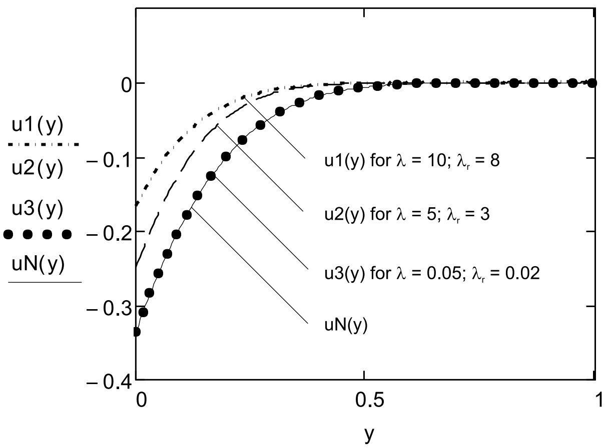

Fig. 1. Variation of the velocity field $u(y, t)$ given by Eq. (11)—curves $u1(y)$, $u2(y)$, $u3(y)$—and Eq. (14)—curve $uN(y)$—for $f = 5$, $\upsilon = 0.003$, $\mu = 2.916$, $t = 10$ s and different values of λ and λ_r.

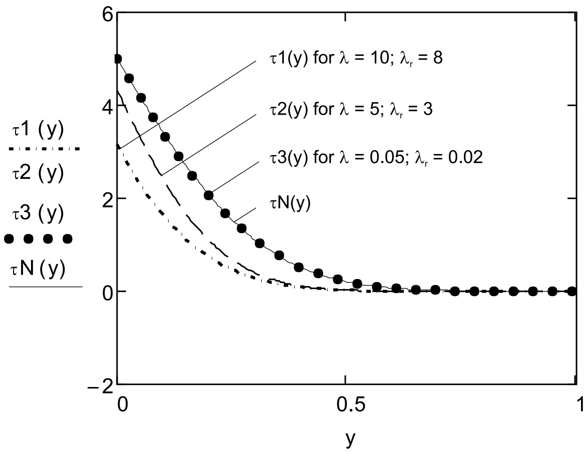

Fig. 2. Variation of the shear stress $\tau(y, t)$ given by Eq. (12)—curves $\tau 1(y)$, $\tau 2(y)$, $\tau 3(y)$—and Eq. (15)—curve $\tau N(y)$—for $f = 5$, $\upsilon = 0.003$, $\mu = 2.916$, $t = 10$ s and different values of λ and λ_r.

used to obtain direct comparisons among the four models. From Eq. (36), for instance, L and Φ decrease for non-Newtonian fluids in comparison with Newtonian fluids (L being considered in absolute value). Furthermore, the same entities decrease for Oldroyd-B fluids compared to Maxwell and second-grade fluids.

The boundary layer thickness δ, as obtained in Eq. (39), increases for second-grade fluids and decreases for Maxwell and Oldroyd-B fluids compared to Newtonian fluids. δ decreases for Oldroyd-B fluids in comparison with second-grade fluids and increases compared to Maxwell fluid. Similar results are obtained from Eq. (40) for the variation of the kinetic energy with time.

In the special cases where λ or $\lambda_r \to 0$, all results that have been obtained tend to those for second-grade or Maxwell fluids performing the same motion. If both times are tending to zero, results similar to those corresponding to Newtonian fluids are recovered. Following Teipel [2], series solutions are established both for the velocity field $u(y, t)$ and for the adequate shear stress $\tau(y, t)$. These solutions, as well as Teipel's solution corresponding to the first problem of Stokes for second-grade fluids, contain two free constants, although all initial and boundary conditions have been satisfied. These constants are determined by means of asymptotic approximations.

Finally, for comparison, the diagrams of the velocity field as well as those for the shear stress corresponding to Newtonian and Oldroyd-B fluids are depicted in Figs. 1 and 2. For small values of λ and λ_r, there is virtually no difference between the two models, their graphs being almost identical. Similar diagrams corresponding to the exact expressions for L, Φ and δ are also presented in Figs. 3–5. The results agree with those obtained from the asymptotic approximations. From these figures, it is clearly seen that L, Φ and δ are smaller for Oldroyd-B fluids compared to Newtonian fluids (L being viewed in absolute value). All these results, as well as the fact that δ, unlike L and Φ, does not depend of the constant shear stress of the plate, are important and have application in engineering.

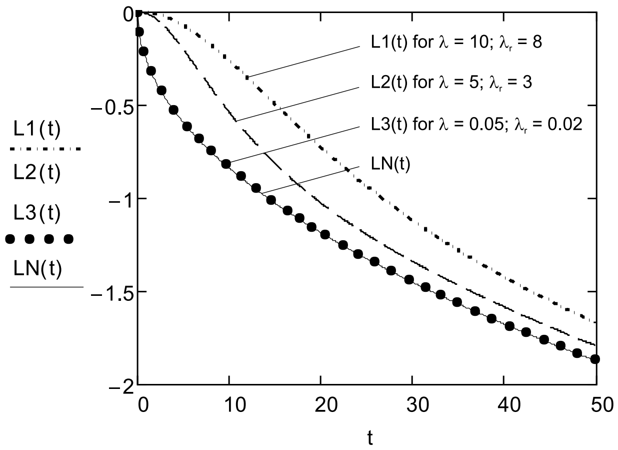

Fig. 3. Variation of the power $L(t)$ given by Eq. (21)—curves $L1(t)$, $L2(t)$, $L3(t)$—and Eq. (24)—curve $LN(t)$—for $f = 5$, $\ell = 0.5$, $v = 0.003$, $\mu = 2.916$ and different values of λ and λ_r.

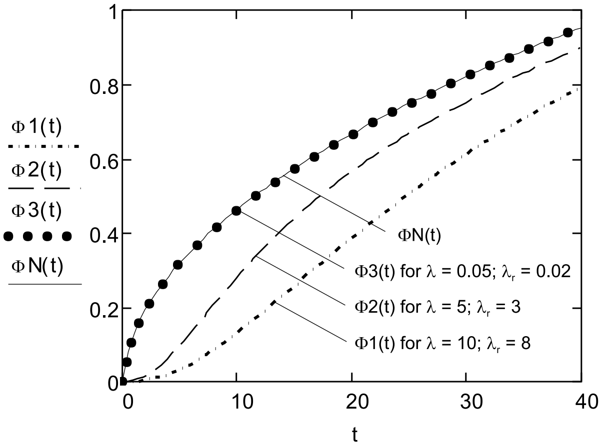

Fig. 4. Variation of the dissipation $\Phi(t)$ given by Eq. (22)—curves $\Phi1(t)$, $\Phi2(t)$, $\Phi3(t)$—and Eq. (25)—curve $\Phi N(t)$—for $f = 5$, $\ell = 0.5$, $v = 0.003$, $\mu = 2.916$ and different values of λ and λ_r.

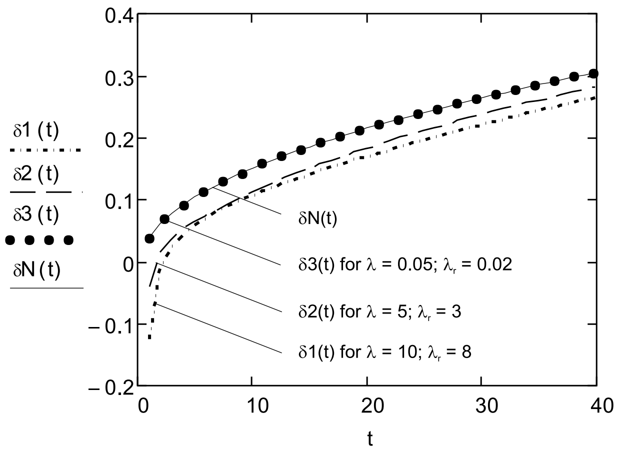

Fig. 5. Variation of the boundary layer thickness $\delta(t)$ given by Eq. (23)—curves $\delta1(t)$, $\delta2(t)$, $\delta3(t)$—and Eq. (26)—curve $\delta N(t)$—for $f = 5$, $\ell = 0.5$, $v = 0.003$, $\mu = 2.916$ and different values of λ and λ_r.

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Appendix

$$\int_0^\infty \left[1 - e^{-\nu t\xi^2}\cos(y\xi)\right]\frac{1}{\xi^2}\,d\xi = \frac{\pi}{2}y + \sqrt{\pi\nu t}\exp\left(-\frac{y^2}{4\nu t}\right) - \frac{\pi}{2}y\,\mathrm{erfc}\left(-\frac{y}{2\sqrt{\nu t}}\right), \tag{A.1}$$

$$\int_0^\infty \frac{\sin(y\xi)}{\xi}e^{-\nu t\xi^2}\,d\xi = \frac{\pi}{2}\,\mathrm{erf}\left(-\frac{y}{2\sqrt{\nu t}}\right), \tag{A.2}$$

$$\int_0^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}, \qquad \int_0^\infty s^2 e^{-s^2}\,ds = \frac{\sqrt{\pi}}{4}, \tag{A.3}$$

$$\int_0^\infty z\,\mathrm{erfc}(z)\,dz = \frac{1}{4}, \tag{A.4}$$

$$\int_0^\infty \left(1 - e^{-\nu t\xi^2}\right)\frac{\sin(y\xi)}{\xi}\,d\xi = \frac{\pi}{2}\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{A.5}$$

$$\int_0^\infty \mathrm{erfc}^2\left(\frac{y}{2\sqrt{\nu t}}\right)dy = 2\left(2 - \sqrt{2}\right)\sqrt{\frac{\nu t}{\pi}}, \tag{A.6}$$

$$\int_0^\infty e^{-a\xi^2}\cos(y\xi)\,d\xi = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-\frac{y^2}{4a}\right), \tag{A.7}$$

$$\int_0^\infty \xi^2 e^{-\nu t\xi^2}\cos(y\xi)\,d\xi = \frac{1}{4\nu t}\sqrt{\frac{\pi}{\nu t}}\left(1 - \frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right), \tag{A.8}$$

$$\int_0^\infty \xi e^{-\nu t\xi^2}\sin(y\xi)\,d\xi = \frac{y}{4\nu t}\sqrt{\frac{\pi}{\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right), \tag{A.9}$$

$$\int_0^\infty \xi^3 e^{-\nu t\xi^2}\sin(y\xi)\,d\xi = \frac{y}{8(\nu t)^2}\sqrt{\frac{\pi}{\nu t}}\left(3 - \frac{y^2}{2\nu t}\right)\exp\left(-\frac{y^2}{4\nu t}\right), \tag{A.10}$$

$$\int_0^\infty y\exp\left(-\frac{y^2}{4\nu t}\right)\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)dy = \left(2 - \sqrt{2}\right)\nu t, \tag{A.11}$$

$$\int_0^\infty y^3\exp\left(-\frac{y^2}{4\nu t}\right)\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)dy = \left(8 - 5\sqrt{2}\right)(\nu t)^2. \tag{A.12}$$

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